Faculty of Science, Technology, Engineering and Mathematics
M337 Complex analysis

M337 Solutions to Practice exam 1

There are alternative solutions to many of these questions. Any correct solution that is set out clearly is worth full marks.

Question 1

(a) (i) $$\frac{3}{\alpha} = \frac{3}{3-3i\sqrt{3}} = \frac{1}{1-i\sqrt{3}} \times \frac{1+i\sqrt{3}}{1+i\sqrt{3}} = \frac{1+i\sqrt{3}}{1^2+(\sqrt{3})^2} = \frac{1+i\sqrt{3}}{4}$$ 2

(ii)

$$\begin{aligned}\alpha^2 &= (3-3i\sqrt{3})^2\\ &= 9 - 18i\sqrt{3} + 9i^2(\sqrt{3})^2\\ &= 9 - 18i\sqrt{3} - 9\times 3\\ &= -18 - 18i\sqrt{3}\end{aligned}$$ 2

(iii) First observe that

$$\begin{aligned}|\alpha^2| &= |-18-18i\sqrt{3}|\\ &= |18(-1-i\sqrt{3})|\\ &= 18|-1-i\sqrt{3}|\\ &= 18\sqrt{(-1)^2+(-\sqrt{3})^2}\\ &= 36.\end{aligned}$$

We also have

$$\operatorname{Arg}(\alpha^2) = \operatorname{Arg}(-18-18i\sqrt{3}) = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}.$$

Then

$$\operatorname{Log}(\alpha^2) = \log|\alpha^2| + i\operatorname{Arg}(\alpha^2) = \log 36 - 2i\pi/3.$$ 3

(b) We have $\alpha = 6e^{-i\pi/3}$. By HB A1 3.2, p17, the fourth roots of α are

$$z_k = 6^{1/4}e^{i(-\pi/12+2\pi k/4)}, \quad \text{for } k = 0,1,2,3.$$

That is,

$$z_0 = 6^{1/4}e^{-i\pi/12}, \quad z_1 = 6^{1/4}e^{5i\pi/12},$$
$$z_2 = 6^{1/4}e^{11i\pi/12}, \quad z_3 = 6^{1/4}e^{17i\pi/12} = 6^{1/4}e^{-7i\pi/12}.$$ 3

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Question 2

(a) (i) We have $\gamma'(t) = 2ie^{it}$. Hence

$$\int_\Gamma \overline{z}\,dz = \int_{-\pi/2}^{0} 2e^{-it} \times 2ie^{it}\,dt$$
$$= 4i[t]_{-\pi/2}^{0} = 2\pi i.$$ 3

(ii) Using the Reverse Contour Theorem, we see that

$$\int_{\widetilde{\Gamma}} \overline{iz}\,dz = -\int_\Gamma \overline{iz}\,dz.$$

Note that $\overline{iz} = -i\overline{z}$. Hence, by part (a)(i),

$$\int_{\widetilde{\Gamma}} \overline{iz}\,dz = -\int_\Gamma -i\overline{z}\,dz = i\int_\Gamma \overline{z}\,dz = i \times 2\pi i = -2\pi.$$ 2

(b) By the Triangle Inequality,

$$|\cosh z| = \left|\tfrac{1}{2}(e^z + e^{-z})\right| \le \tfrac{1}{2}(|e^z| + |e^{-z}|).$$

Let $z = x + iy$. Then $|e^z| = |e^{x+iy}| = |e^x||e^{iy}| = e^x$ and $|e^{-z}| = e^{-x}$. If z belongs to $C = \{z : |z| = 3\}$, then $x \le 3$, so

$$|\cosh z| \le \tfrac{1}{2}(e^x + e^{-x}) = \cosh x \le \cosh 3.$$

Next, for $z \in C$, we can use the backwards form of the Triangle Inequality to give

$$|z^4 + 12| \ge |z^4| - |12| = |z|^4 - 12 = 81 - 12 = 69.$$

Thus, for $z \in C$,

$$\left|\frac{7\cosh z}{z^4 + 12}\right| \le \frac{7\cosh 3}{69}.$$

Since the function $f(z) = (7\cosh z)/(z^4 + 12)$ is continuous on the circle C, which has length 6π, we can apply the Estimation Theorem to give

$$\left|\int_C \frac{7\cosh z}{z^4 + 12}\,dz\right| \le \frac{7\cosh 3}{69} \times 6\pi = \frac{14\pi\cosh 3}{23}.$$ 5

10 Total

Question 3

(a) Note that $z^3 + z = z(z^2 + 1)$. So the function f has a simple pole at each of the three solutions 0 and $\pm i$ of $z^3 + z = 0$.

Using the Cover-up Rule, we obtain

$$\operatorname{Res}(f, 0) = \frac{0+4}{0^2+1} = 4,$$

$$\operatorname{Res}(f, i) = \frac{i+4}{i(i+i)} = \frac{i+4}{-2} = -2 - \tfrac{1}{2}i,$$

$$\operatorname{Res}(f, -i) = \frac{-i+4}{-i(-i-i)} = \frac{-i+4}{-2} = -2 + \tfrac{1}{2}i.$$ 4

(b) Let $p(t) = t + 4$ and $q(t) = t^3 + t$. Then the degree of q exceeds that of p by 2 and, by part (a), the poles of $f = p/q$ on the real axis are simple. Hence we can apply HB C1 3.9, p62, with $k = 2$, to see that

$$\int_{-\infty}^{\infty} \frac{t+4}{t^3+t} e^{2it}\, dt = 2\pi i S + \pi i\, T,$$

where S is the sum of the residues of the function

$$g(z) = \frac{z+4}{z^3+z} e^{ikz}$$

at the poles in the upper half-plane, and T is the sum of the residues of g at the poles on the real axis. Using the residues found in part (a) we see that

$$\operatorname{Res}(g, 0) = \operatorname{Res}(f, 0) \times e^{2i\times 0} = 4, \quad \text{and}$$

$$\operatorname{Res}(g, i) = \operatorname{Res}(f, i) \times e^{2i\times i} = (-2 - \tfrac{1}{2}i)e^{-2}.$$

So we have that

$$\int_{-\infty}^{\infty} \frac{t+4}{t^3+t} e^{2it}\, dt = 2\pi i e^{-2}(-2 - \tfrac{1}{2}i) + \pi i \times 4 = \pi e^{-2} + (4\pi - 4\pi e^{-2})i.$$

Therefore, using HB C1 3.10, p62, we can equate real and imaginary parts to obtain

$$\int_{-\infty}^{\infty} \frac{t+4}{t^3+t} \cos 2t\, dt = \pi e^{-2} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{t+4}{t^3+t} \sin 2t\, dt = 4\pi - 4\pi e^{-2}.$$ 6

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Question 4

(a) By HB C3 3.8, p76, $\beta = i$ (the centre of C) is the unique inverse point of ∞ with respect to C. 2

(b) We apply the inverse points method for finding images of generalised circles under Möbius transformations. Observe that

$$f(i) = \frac{i+i}{i \times i + 2} = \frac{2i}{1} = 2i \quad \text{and} \quad f(\infty) = \frac{1}{i} = -i.$$

It follows that $2i$ and $-i$ are inverse points with respect to $f(C)$, so $f(C)$ has an equation of the form

$$|z - 2i| = k|z + i|, \quad \text{for some } k > 0.$$

Now, $-i \in C$, so

$$f(-i) = 0 \in f(C).$$

Hence

$$k = \frac{|0 - 2i|}{|0 + i|} = \frac{2}{1} = 2.$$

Therefore $f(C)$ has equation

$$|z - 2i| = 2|z + i|$$

in Apollonian form. 5

(c) By HB C3 3.11, p76, with $\alpha = 2i$, $\beta = -i$ and $k = 2$, the centre of $f(C)$ is

$$\lambda = \frac{2i - 2^2 \times (-i)}{1 - 2^2} = -2i$$

and the radius is

$$r = \frac{2|2i - (-i)|}{|1 - 2^2|} = 2.$$ 3

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Question 5

(a) The conjugate velocity function is

$$\overline{q}(z) = \frac{-iz^2}{z-i}.$$

Let Γ denote the circle $\{z : |z| = 2\}$. The Circulation and Flux Contour Integral tells us that

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \frac{-iz^2}{z-i}\,dz.$$

We can evaluate this integral using Cauchy's Integral Formula (HB B2 2.1, p45). Let $f(z) = -iz^2$. Then f is analytic on the simply connected region $\mathbb{C}$, and Γ is a simple-closed contour in $\mathbb{C}$. Then by Cauchy's Integral Formula,

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = 2\pi i f(i) = 2\pi i \times i = -2\pi.$$

Therefore $\mathcal{F}_\Gamma = 0$, so i is not a sink nor a source, and $\mathcal{C}_\Gamma = -2\pi < 0$, so i is a clockwise vortex. 5

(b) Let $\mathcal{S} = \{z : |z| > 1\}$. By HB D1 3.2, p85, the Joukowski function J is a one-to-one conformal mapping from $\mathcal{S}$ onto $\mathbb{C} - [-2, 2]$. So we seek a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$, which we will compose with J. Using the table of standard conformal mappings (HB C3 4.10, p78), we consider the Möbius transformation

$$h(z) = \frac{z-1}{z+1}.$$

We shall prove that h is a one-to-one conformal mapping from $\mathcal{R}$ to $\mathcal{S}$.

To see that h preserves the correct orientation, consider the point $-2 \in \mathcal{R}$. We have that $h(-2) = 3 \in \mathcal{S}$, so the *left* half-plane $\mathcal{R} \cup \{-1\}$ is mapped onto the *outside* of the unit circle $\mathcal{S} \cup \{\infty\}$.

Note also that $h(-1) = \infty$. Since Möbius transformations are one-to-one conformal mappings on $\widehat{\mathbb{C}}$, the element -1 is the only element of $\widehat{\mathbb{C}}$ which maps to ∞. Therefore, by restricting the domain of h to $\mathcal{R}$, we see that h is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

It follows that the composite mapping

$$g(z) = J(h(z)) = \frac{z-1}{z+1} + \frac{z+1}{z-1} = \frac{(z-1)^2 + (z+1)^2}{z^2-1} = \frac{2(z^2+1)}{z^2-1},$$

is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathbb{C} - [-2, 2]$. It is one-to-one and conformal because it is a composite of one-to-one conformal mappings on their respective domains. 5

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Question 6

(a) By HB D2 2.1, p89, the iteration sequence

$$z_{n+1} = (iz_n - 1)(z_n + 2i) = iz_n^2 - 3z_n - 2i, \quad n = 0, 1, 2, \ldots,$$

is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$

where

$$d = i \times (-2i) + \tfrac{1}{2} \times (-3) - \tfrac{1}{4} \times (-3)^2 = 2 - \tfrac{3}{2} - \tfrac{9}{4} = -\tfrac{7}{4}.$$

The conjugating function is

$$h(z) = iz - \tfrac{3}{2}.$$

Hence

$$w_0 = h(z_0) = i \times 0 - \tfrac{3}{2} = -\tfrac{3}{2}.$$ 4

(b) (i) Let $c = -1 + \frac{1}{5}i$. Observe that

$$|c+1| = \left|\tfrac{1}{5}i\right| = \tfrac{1}{5} < \tfrac{1}{4}.$$

So by HB D2 4.11(b), p92, the function P_c has an attracting 2-cycle. Hence $-1 + \frac{1}{5}i \in M$, by HB D2 4.10, p92. 3

(ii) Let $c = \frac{1}{5} + \frac{2}{5}i$. Observe that

$$\begin{aligned}
&\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c \\
=&\left(8\left|\tfrac{1}{5} + \tfrac{2}{5}i\right|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\left(\tfrac{1}{5} + \tfrac{2}{5}i\right) \\
=&\left(8\left(\tfrac{1}{25} + \tfrac{4}{25}\right) - \tfrac{3}{2}\right)^2 + \tfrac{8}{5} \\
=&\left(\tfrac{40}{25} - \tfrac{3}{2}\right)^2 + \tfrac{8}{5} \\
=&\left(\tfrac{1}{10}\right)^2 + \tfrac{8}{5} \\
=&\tfrac{1}{100} + \tfrac{8}{5}.
\end{aligned}$$

Since $\frac{1}{100} + \frac{8}{5} < 3$, we see from HB D2 4.11(a), p92, that the function P_c has an attracting fixed point. Hence $\frac{1}{5} + \frac{2}{5}i \in M$, by HB D2 4.10, p92. 3

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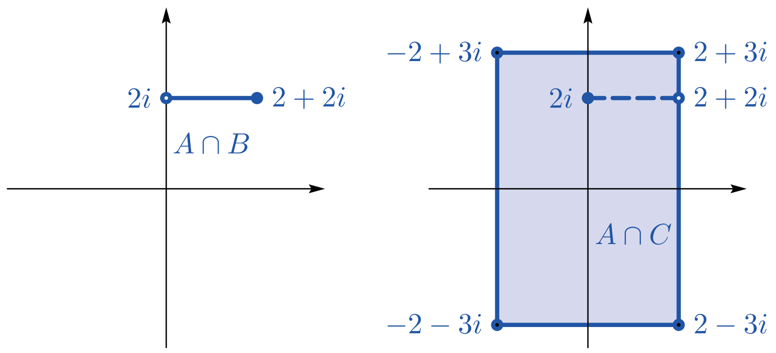
Question 7

(a) (i)



4

(ii)
- The set A is closed and bounded, so it is compact. The function f is continuous on $\mathbb{C}$, so it is continuous on A. Hence f is bounded on A, by the Boundedness Theorem, HB A3 5.11, p35.
- For any point $z \in B$, we have $z = x + 2i$ for some $x \in \mathbb{R}$.
  So $f(z) = \sin(z - 2i) = \sin(x + 2i - 2i) = \sin x$.
  We know that $|\sin x| \leq 1$ for $x \in \mathbb{R}$, so $|f(z)| \leq 1$ for all $z \in B$.
  Hence f is bounded on B.
- Suppose that f is bounded on C.
  Since f is bounded on B, and $B \cup C = \mathbb{C}$, then f must be bounded on all of $\mathbb{C}$.
  Note also that $f(z) = \sin(z - 2i)$ is an entire function.
  Since f is both bounded and entire, by Liouville's Theorem, HB B2 2.2, p45, f is a constant function.
  But $f(2i) = 0$ and $f(\pi/2 + 2i) = 1$, so f cannot be a constant function. This is a contradiction.
  Hence f is not bounded on C. 6

(b) Let $z = x + iy$. Then

$$f(z) = x\exp(x-iy) = xe^x e^{-iy} = xe^x(\cos(-y)+i\sin(-y)) = xe^x(\cos y - i\sin y).$$

Define

$$u(x,y) = xe^x \cos y \quad \text{and} \quad v(x,y) = -xe^x \sin y.$$

Then $f(z) = u(x,y) + iv(x,y)$, and

$$\frac{\partial u}{\partial x}(x,y) = xe^x \cos y + e^x \cos y = (x+1)e^x \cos y,$$

$$\frac{\partial u}{\partial y}(x,y) = -xe^x \sin y,$$

$$\frac{\partial v}{\partial x}(x,y) = -xe^x \sin y - e^x \sin y = -(x+1)e^x \sin y,$$

$$\frac{\partial v}{\partial y}(x,y) = -xe^x \cos y.$$

The first Cauchy–Riemann equation is

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \iff (x+1)e^x \cos y = -xe^x \cos y \iff (2x+1)e^x \cos y = 0.$$

Since $e^x \neq 0$, this equation has solutions $x = -\frac{1}{2}$ and $y = (n + \frac{1}{2})\pi$, for $n \in \mathbb{Z}$.

The second Cauchy–Riemann equation is

$$\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \iff -xe^x \sin y = (x{+}1)e^x \sin y \iff (2x{+}1)e^x \sin y = 0.$$

The solutions of this equation are $x = -\frac{1}{2}$ and $y = n\pi$, for $n \in \mathbb{Z}$.

Hence both the Cauchy–Riemann equations are satisfied if and only if $x = -\frac{1}{2}$. Hence the Cauchy–Riemann equations are satisfied if and only if z has the form $z = -\frac{1}{2} + iy$ for any $y \in \mathbb{R}$.

Since the partial derivatives exist and are continuous on $\mathbb{C}$, and the Cauchy–Riemann equations are satisfied at $z = -\frac{1}{2} + iy$, $y \in \mathbb{R}$, we see from the Cauchy–Riemann Converse Theorem that f is differentiable at all these points.

Since the Cauchy–Riemann equations are not satisfied at other points, the Cauchy–Riemann Theorem tells us that f is not differentiable at any other points of $\mathbb{C} - \{-\frac{1}{2} + iy : y \in \mathbb{R}\}$.

At the point $z = -\frac{1}{2}$, we have

$$\begin{aligned} f'(-\tfrac{1}{2}) &= \frac{\partial u}{\partial x}(-\tfrac{1}{2}, 0) + i\frac{\partial v}{\partial x}(-\tfrac{1}{2}, 0) \\ &= (-\tfrac{1}{2} + 1)e^{-\frac{1}{2}} \cos 0 + i \times 0 \\ &= \tfrac{1}{2}e^{-\frac{1}{2}}. \end{aligned}$$

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Question 8

(a) (i) By the Radius of Convergence Formula, the radius of convergence is

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{i^n (n+1)!/5^n}{i^{n+1}(n+2)!/5^{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{5}{i} \times \frac{1}{n+2} \right| \\ &= \lim_{n\to\infty} \frac{5}{n+2} = 0. \end{aligned}$$

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(ii) By the Radius of Convergence Formula, the radius of convergence is

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{3n^2 - 5n + e^{in}}{3(n+1)^2 - 5(n+1) + e^{i(n+1)}} \right| \\ &= \lim_{n\to\infty} \left| \frac{3n^2 - 5n + e^{in}}{3n^2 + n - 2 + e^i e^{in}} \right| \\ &= \lim_{n\to\infty} \left| \frac{3 - 5/n + e^{in}/n^2}{3 + 1/n - 2/n^2 + e^i e^{in}/n^2} \right|. \end{aligned}$$

Now $|e^{in}| = 1$, for $n = 1, 2, \ldots$, hence

$$e^{in}/n^2 \to 0 \quad \text{and} \quad e^i e^{in}/n^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore $R = 3/3 = 1$.

3

(b) (i) We have

$$\sinh z = z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots, \quad \text{for } z \in \mathbb{C},$$

$$\cos w = 1 - \frac{1}{2!}w^2 + \frac{1}{4!}w^4 - \cdots, \quad \text{for } w \in \mathbb{C}.$$

Let $w = \sinh z$. Since $\sinh 0 = 0$, we can apply the Composition Rule for Power Series to give

$$\begin{aligned}\cos(\sinh z) &= 1 - \frac{1}{2}\left(z + \frac{1}{3!}z^3 + \cdots\right)^2 + \frac{1}{24}\left(z + \frac{1}{3!}z^3 + \cdots\right)^4 - \cdots \\ &= 1 - \frac{1}{2}\left(z^2 + \frac{1}{3}z^4 + \cdots\right) + \frac{1}{24}\left(z^4 + \cdots\right) + \cdots \\ &= 1 - \frac{1}{2}z^2 - \frac{1}{6}z^4 + \frac{1}{24}z^4 + \cdots \\ &= 1 - \frac{1}{2}z^2 - \frac{1}{8}z^4 + \cdots.\end{aligned}$$

Since g is an entire function, this Taylor series converges to $g(z)$ for each $z \in \mathbb{C}$, by HB B3 3.5, p51. 5

(ii) The function $f(z) = zg(3/z)$ is analytic on the simply connected region $\mathbb{C}$ except for a singularity at 0. By part (b)(i) we have

$$\begin{aligned}zg(3/z) &= z\left(1 - \frac{1}{2}\left(\frac{3}{z}\right)^2 - \frac{1}{8}\left(\frac{3}{z}\right)^4 + \cdots\right) \\ &= z - \frac{9}{2z} - \frac{81}{8z^3} + \cdots,\end{aligned}$$

for $z \in \mathbb{C} - \{0\}$. Hence

$$\operatorname{Res}(f, 0) = -\frac{9}{2}.$$

Applying the Residue Theorem we see that

$$\int_C zg(3/z)\,dz = 2\pi i \times \left(-\frac{9}{2}\right) = -9\pi i.$$ 4

(c) We are given that f is entire and satisfies the property

$$f\left(\frac{1}{3^n}\right) = \frac{1}{3^{n+1}} \qquad (\star)$$

for all $n \in \mathbb{Z}$.

The function $g(z) = \frac{1}{3}z$ is entire, and satisfies property $(\star)$, since

$$g\left(\frac{1}{3^n}\right) = \frac{1}{3} \times \frac{1}{3^n} = \frac{1}{3^{n+1}}$$

for all $n \in \mathbb{Z}$.

We prove that g is the only entire function with property $(\star)$ by using the Uniqueness Theorem, HB B3 4.8, p54.

Let $S = \left\{\frac{1}{3^n} : n \in \mathbb{Z}\right\}$, and let $\mathcal{R} = \mathbb{C}$.

The sequence $\left(\frac{1}{3^n}\right)$ for $n = 1, 2, 3, \ldots$ is contained in S. The limit of the sequence is $0 \in \mathcal{R}$, so S has a limit point in $\mathcal{R}$.

Since f and g are both entire functions, they are analytic on the region $\mathcal{R} = \mathbb{C}$, and since they both satisfy property $(\star)$, they agree on the set S.

Therefore by the Uniqueness Theorem, f and g agree throughout $\mathcal{R} = \mathbb{C}$, and so $g(z) = \frac{1}{3}z$ is the unique entire function with property $(\star)$.

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Question 9

(a) (i) Observe that

$$\begin{aligned}|\sin z| &= \left| z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right| \\ &\leq |z| + \left|\frac{z^3}{3!}\right| + \left|\frac{z^5}{5!}\right| + \cdots,\end{aligned}$$

using the Triangle Inequality for Series. So if $|z| = 1$, then

$$|\sin z| \leq 1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots = \sinh 1.$$

Now

$$\sinh 1 = \tfrac{1}{2}(e - e^{-1}) < \tfrac{1}{2}(3 - 0) = \tfrac{3}{2}.$$

Hence $|\sin z| < \frac{3}{2}$, for $|z| = 1$. 4

(ii) Let $f(z) = 2z + \sin z$. We must find all solutions of the equation $f(z) = 0$ in the open unit disc $\{z : |z| < 1\}$. One solution is $z = 0$, because

$$f(0) = 2 \times 0 + \sin 0 = 0.$$

Next we define $g(z) = 2z$. If $|z| = 1$, then

$$|f(z) - g(z)| = |\sin z| < \tfrac{3}{2},$$

by part (a)(i). Also, for $|z| = 1$, we have $|g(z)| = 2|z| = 2$. Hence

$$|f(z) - g(z)| < |g(z)|, \quad \text{for } |z| = 1.$$

Now f and g are analytic on the simply connected region $\mathbb{C}$, and $\{z : |z| = 1\}$ is a simple-closed contour in $\mathbb{C}$, so we see from Rouché's Theorem that f has the same number of zeros as g inside $\{z : |z| = 1\}$, namely 1.

It follows that $z = 0$ is the only solution of the equation $2z + \sin z = 0$ in the open unit disc $\{z : |z| < 1\}$. 6

(b) Let $f(z) = z\exp(iz^3 - 2)$ and $\mathcal{R} = \{z : |z| < 3\}$. Then f is analytic on $\mathbb{C}$, so it is analytic (and non-constant) on $\mathcal{R}$ and continuous on $\overline{\mathcal{R}} = \{z : |z| \leq 3\}$. We can therefore apply the Maximum Principle to see that the maximum value of $|f(z)|$ on $\overline{\mathcal{R}}$ is attained on the boundary $\partial\mathcal{R}$ and is not attained in $\mathcal{R}$. Hence

$$\max\{|f(z)| : |z| \leq 3\} = \max\{|f(z)| : |z| = 3\}.$$

Now, if $|z| = 3$, then $z = 3e^{it}$, where $0 \leq t < 2\pi$. Hence

$$\begin{aligned}
|f(z)| &= |z\exp(iz^3 - 2)| \\
&= 3|\exp(27ie^{3it} - 2)| \\
&= 3|\exp(27i\cos 3t - 27\sin 3t - 2)| \\
&= 3|\exp(27i\cos 3t)|\,|\exp(-27\sin 3t - 2)| \\
&= 3\exp(-27\sin 3t - 2).
\end{aligned}$$

Since $x \longmapsto e^x$ is an increasing real function, the expression $3\exp(-27\sin 3t - 2)$ takes its maximum value when $\sin 3t = -1$. This happens when (and only when) $t = \pi/2, 7\pi/6, 11\pi/6$, corresponding to the values

$$z = 3e^{i\pi/2} = 3i, \quad z = 3e^{7i\pi/6} = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i \quad \text{and} \quad z = 3e^{11i\pi/6} = \frac{3\sqrt{3}}{2} - \frac{3}{2}i.$$

At these values,

$$|f(z)| = 3\exp(27 - 2) = 3e^{25}.$$

In summary, then,

$$\max\{|z\exp(iz^3 - 2)| : |z| \leq 3\} = 3e^{25},$$

and this maximum is attained at the points $z = 3i, \pm\frac{3\sqrt{3}}{2} - \frac{3}{2}i$ only. 10

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